ON FUNCTION THEORY IN QUANTUM DISC: A q-ANALOGUE OF BEREZIN TRANSFORM

D. Shklyarov *S. Sinel'shchikov* *L. Vaksman*

Institute for Low Temperature Physics & Engineering
National Academy of Sciences of Ukraine

Let α be a positive number.

Section 1 of this work contains a study of Toeplitz-Bergman operators with finite symbols in the quantum disc, and section 4 deals already with Toeplitz-Bergman operators with bounded symbols. An alternate way of producing Toeplitz-Bergman operators with polynomial symbols is described in section 7 (lemma 7.2).

Section 2 introduces a Berezin transform $B_{q,\alpha}$ for finite functions in the quantum disc; the same is done in section 4 for bounded functions. An alternate way of constructing a Berezin transform for a polynomial function is described in sections 6, 7 (proposition 6.6 and lemma 7.2).

An asymptotic expansion (3.2), (3.6) for a Berezin transform for a finite function is obtained in section 3; a similar expansion (5.2) for the case of a bounded function can be found in section 5. An application of the latter result to formal series with polynomial coefficients in section 8 affords the main result of this work (theorem 8.4).

We use the background and notation used in [8, 9, 10].

1 Toeplitz-Bergman operators with finite symbols

Consider the covariant algebra $\mathbb{C}[z]_q$ (see [9]). Algebraically it is isomorphic to the polynomial algebra $\mathbb{C}[z]$, and the $U_q\mathfrak{sl}_2$-action is determined by the relations

$$K^{\pm 1}z=q^{\pm 2}z, \qquad Fz=q^{1/2}.$$

We also follow [12] in using a covariant (left) $\mathbb{C}[z]_q$-module with the generator $\mathbb{I}$ and the relations

$$K^{\pm 1}\mathbb{I}=q^{\pm(2\alpha+1)}\mathbb{I}, \qquad F\mathbb{I}=0.$$

Denote this covariant module by $\mathbb{C}[z]_{q,\alpha}$

Let $F_{q,\alpha}\subset \mathrm{End}(\mathbb{C}[z]_{q,\alpha})$ be the covariant algebra of linear operators $A: z^j\mapsto \sum\limits_{m\in\mathbb{Z}_+} a_{mj}z^m$, $j\in\mathbb{Z}_+$, with finitely many nonzero matrix elements a_{mj}. In virtue of this definition, $F_{q,\alpha}\hookrightarrow \mathbb{C}[z]_{q,\alpha}\otimes\mathbb{C}[z]^*_{q,\alpha}$

Our immediate purpose is to construct a morphism of $U_q\mathfrak{sl}_2$-modules $D(U)_q\to F_{q,\alpha}$ which is normally called a Toeplitz quantization.

Remind the notation (see [8]):

$$\int\limits_{U_q} f d\nu_\alpha = \frac{1-q^{4\alpha}}{1-q^2}\int\limits_{U_q} f(1-zz^*)^{2\alpha+1}d\nu,$$

$$(f_1,f_2)_{q,\alpha} = \int\limits_{U_q} f_2^* f_1 d\nu_\alpha. \tag{1.1}$$

Form a completion of the linear space $D(U)_q$ with respect to the norm $\|f\|_{q,\alpha} = (f,f)_{q,\alpha}^{1/2}$. It is easy to show that this Hilbert space admits an embedding into $D(U)_q'$ and is canonically isomorphic to the space $L^2_{q,\alpha}$ defined in [8].

Let $P_{q,\alpha}$ be the orthogonal projection in $L^2_{q,\alpha}$ onto the closure $H^2_{q,\alpha}$ of the subspace $\mathbb{C}[z]_{q,\alpha} \subset L^2_{q,\alpha}$. Given $\overset{\circ}{f} \in D(U)_q$, we call the linear operator

$$\widehat{f} : \mathbb{C}[z]_{q,\alpha} \to \mathbb{C}[z]_{q,\alpha}; \qquad \widehat{f} : \psi \mapsto P_{q,\alpha}(\overset{\circ}{f}\ \psi), \qquad \psi \in \mathbb{C}[z]_{q,\alpha}$$

a Toeplitz-Bergman operator with the finite symbol $\overset{\circ}{f}$. This is well defined, as one can see from

Proposition 1.1 *With $\overset{\circ}{f} \in D(U)_q$, for all but finitely many $m, j \in \mathbb{Z}_+$ the integral $I_{m,j} = \int\limits_{U_q} z^{*m}\ \overset{\circ}{f}\ z^j d\nu_\alpha$ is zero.*

Proof. It was shown in [8] that for any $\overset{\circ}{f} \in D(U)_q$ one has $z^{*N}\ \overset{\circ}{f} = \overset{\circ}{f}\ z^N = 0$ for some $N \in \mathbb{N}$. Hence $I_{mj} = 0$ if $\max(m,j) \geq N$. □

A straightforward consequence of proposition 1.1 is that the Toeplitz-Bergman operator with a finite symbol belongs to the covariant algebra $F_{q,\alpha}$.

Proposition 1.2 *Toeplitz quantization $D(U)_q \to F_{q,\alpha}$, $\overset{\circ}{f} \mapsto \widehat{f}$, is a morphism of $U_q\mathfrak{sl}_2$-modules.*

Proof. One can deduce from the invariance of the scalar product in $\mathbb{C}[z]_{q,\alpha}$ and the covariance of the left $\mathbb{C}[z]_q$-module $\mathbb{C}[z]_{q,\alpha}$ that the linear map

$$D(U)_q \otimes \mathbb{C}[z]_{q,\alpha} \to \mathbb{C}[z]_{q,\alpha}; \qquad \overset{\circ}{f} \otimes \psi \mapsto P_{q,\alpha}(\overset{\circ}{f}\ \psi)$$

is a morphism of $U_q\mathfrak{sl}_2$-modules. On the other hand, we need to demonstrate that the linear map

$$D(U)_q \to \mathbb{C}[z]_{q,\alpha} \otimes \mathbb{C}[z]^*_{q,\alpha}, \qquad \overset{\circ}{f} \mapsto \widehat{f}$$

(the tensor product here requires no completion due to proposition 1.1). Observe that the two statements are equivalent to $U_q\mathfrak{sl}_2$-invariance of the same element of the corresponding completion of the tensor product $F_{q,\alpha} \otimes D(U)_q'$, which is determined by the canonical isomorphisms $\mathrm{End}_{\mathbb{C}}(V_1,V_2) \simeq V_2 \widehat{\otimes} V_1^*$, $(V_1 \otimes V_2)^* \simeq V_2^* \widehat{\otimes} V_1^*$. □

Remind the notation $\mathrm{Fun}(U)_q = \mathrm{Pol}(\mathbb{C})_q + D(U)_q$. A very important construction of [8] was the representation T of $\mathrm{Fun}(U)_q$ in the infinitely dimensional vector space H. A basis in H was formed by the vectors $v_j = T(z^j)v_0$, $j \in \mathbb{Z}_+$ (see [8]). T provides a one-to-one map between the space of finite functions $D(U)_q$ and the space of linear operators in H whose matrices in the basis $\{v_j\}_{j\in\mathbb{Z}_+}$ have finitely many non-zero entries. For $j \in \mathbb{Z}_+$, let f_j stand for such finite function that $T(f_j)v_k = \delta_{jk}v_k$, $k \in \mathbb{Z}_+$.

The relation $(1 - zz^*)v_j = q^{2j}v_j$, $j \in \mathbb{Z}_+$, motivates the following definition:

$$(1 - zz^*)^\lambda \overset{def}{=} \sum_{n=0}^{\infty} q^{2n\lambda} f_n, \qquad \lambda \in \mathbb{C}.$$

(The series converges in the topological space $D(U)'_q$.)

The work [9] presents an explicit form of the invariant integral in the quantum disc. It is easy to show that for any finite function f

$$\int\limits_{U_q} f d\nu = (1 - q^2)\,\mathrm{tr}\, T(f(1 - zz^*)^{-1}).$$

Remark 1.3. Let $\widehat{f_0}$ be the Toeplitz-Bergman operator with symbol f_0. It follows from the relations $z^* f_0 = f_0 z = 0$, $\int\limits_{U_q} f_0 d\nu = 1 - q^2$ that

$$\widehat{f_0} : z^j \mapsto \begin{cases} 1 - q^{4\alpha} & , \quad j = 0 \\ 0 & , \quad j \neq 0 \end{cases}.$$

Now the relation (1.1), the trace properties and the definition of $D(U)'_q$ imply

Lemma 1.4

1. *For all $f \in D(U)_q$, $\lambda \in \mathbb{C}$*

$$\int\limits_{U_q} f(z)(1 - zz^*)^\lambda d\nu = \int\limits_{U_q} (1 - zz^*)^\lambda f(z) d\nu,$$

2.

$$\int\limits_{U_q} f_1(z) f_2(z)(1 - zz^*) d\nu(z) = \int\limits_{U_q} f_2(z) f_1(z)(1 - zz^*) d\nu(z)$$

for all $f_1(z) \in D(U)'_q$, $f_2(z) \in D(U)_q$.

The following proposition describes an integral representation for matrix elements of Toeplitz-Bergman operator.

Proposition 1.5 *Let* $\overset{\circ}{f} \in D(U)_q$ *and* $\widehat{f} : \mathbb{C}[z]_{q,\alpha} \to \mathbb{C}[z]_{q,\alpha}$; $\widehat{f} : z^j \mapsto \sum\limits_{m \in \mathbb{Z}_+} \widehat{f}_{mj} z^m$ *be a Toeplitz-Bergman operator with symbol* $\overset{\circ}{f}$. *Then*

$$\widehat{f}_{mj} = \frac{1-q^{4\alpha}}{1-q^2} \int\limits_{U_q} P_{z,mj} \overset{\circ}{f}(z) d\nu(z), \tag{1.2}$$

with

$$P_{z,mj} = \frac{(q^{4\alpha+2};q^2)_m}{(q^2;q^2)_m} q^{2j} z^j (1-zz^*)^{2\alpha+1} z^{*m}. \tag{1.3}$$

Proof. Apply the relation [5, 8]

$$(z^m, z^l)_{q,\alpha} = \frac{(q^2;q^2)_m}{(q^{4\alpha+2};q^2)_m} \delta_{ml}, \qquad m, l \in \mathbb{Z}_+$$

to get

$$\widehat{f}_{mj} = \frac{(\overset{\circ}{f} z^j, z^m)_{q,\alpha}}{(z^m, z^m)_{q,\alpha}} = \frac{(q^{4\alpha+2};q^2)_m}{(q^2;q^2)_m} \int\limits_{U_q} z^{*m} \overset{\circ}{f} z^j d\nu_\alpha =$$

$$= \frac{1-q^{4\alpha}}{1-q^2} \cdot \frac{(q^{4\alpha+2};q^2)_m}{(q^2;q^2)_m} \int\limits_{U_q} z^{*m} \overset{\circ}{f} z^j (1-zz^*)^{2\alpha+1} d\nu.$$

Hence, by lemma 1.4,

$$\widehat{f}_{mj} = \frac{1-q^{4\alpha}}{1-q^2} \cdot \frac{(q^{4\alpha+2};q^2)_m}{(q^2;q^2)_m} \int\limits_{U_q} (1-zz^*) z^j (1-zz^*)^{2\alpha} z^{*m} \overset{\circ}{f} d\nu. \tag{1.4}$$

It remains to apply the relation

$$(1-zz^*)z = q^2 z(1-zz^*). \qquad \square$$

Remark 1.6. The matrix $P_z = (P_{z,mj})_{m,j \in \mathbb{Z}_+}$ is a q-analogue for the matrix of a one-dimensional orthogonal projection onto the subspace generated by the vector k_z from an overfull system (see [1]). A q-analogue of the overfull system itself is presented in the Appendix.

Remark 1.7. It follows from proposition 1.2 and the relation $U_q\mathfrak{sl}_2 \cdot \widehat{f}_0 = F_{q,\alpha}$ to be proved later on (see proposition 6.4) that the map $D(U)_q \to F_{q,\alpha}$, $\overset{\circ}{f} \mapsto \widehat{f}$, given by Toeplitz quantization is onto.

2 Berezin transform: finite functions

Consider a $U_q\mathfrak{sl}_2$-module V and the covariant algebra $\mathrm{End}_{\mathbb{C}}(V)_f \simeq V \otimes V^*$. There is a well known (see [2, 9]) formula for an invariant integral

$$\mathrm{tr}_q : \mathrm{End}_{\mathbb{C}}(V)_f \to \mathbb{C}, \qquad \mathrm{tr}_q : A \mapsto \mathrm{tr}(A \cdot K^{-1}).$$

In the case $V = \mathbb{C}[z]_{q,\alpha}$ and $A : z^j \mapsto \sum\limits_{m \in \mathbb{Z}_+} a_{mj} z^m$ being an element of the covariant algebra $F_{q,\alpha} \subset \mathrm{End}_{\mathbb{C}}(\mathbb{C}[z]_{q,\alpha})_f$, one has $\mathrm{tr}_q(A) = \sum\limits_{k \in \mathbb{Z}_+} a_{kk} q^{-2k}$.

Given a linear operator $\widehat{f} \in F_{q,\alpha}$, a distribution $f \in D(U)'_q$ is said to be a symbol of $\widehat{f}$ if for all $\overset{\circ}{\psi} \in D(U)_q$

$$\int\limits_{U_q} f \cdot \overset{\circ}{\psi} \, d\nu = \frac{1-q^2}{1-q^{4\alpha}} \mathrm{tr}_q(\widehat{f} \widehat{\psi}). \tag{2.1}$$

(Here $\widehat{\psi}$ is the Toeplitz-Bergman operator with symbol $\overset{\circ}{\psi}$.)

This definition is a q-analogue of the Berezin's definition, as one can observe from relation (3.15) from [1].

Proposition 2.1 *The covariant symbol of a linear operator $\widehat{f} : z^j \mapsto \sum\limits_{m \in \mathbb{Z}_+} \widehat{f}_{mj} z^m$, $j \in \mathbb{Z}_+$, from the algebra $F_{q,\alpha}$, is given by*

$$f = \mathrm{tr}_q(\widehat{f} \cdot P_z) = \sum_{j,m \in \mathbb{Z}_+} \widehat{f}_{jm} P_{z,mj} q^{-2j}. \tag{2.2}$$

Proof. By a virtue of (1.2)

$$\mathrm{tr}_q(\widehat{f} \widehat{\psi}) = \sum_{j,m \in \mathbb{Z}_+} \widehat{f}_{jm} \widehat{\psi}_{mj} q^{-2j} = \sum_{j,m \in \mathbb{Z}_+} \widehat{f}_{jm} \frac{1-q^{4\alpha}}{1-q^2} \int\limits_{U_q} P_{z,mj} \overset{\circ}{\psi} \, d\nu(z) \cdot q^{-2j} =$$

$$= \int\limits_{U_q} \left(\frac{1-q^{4\alpha}}{1-q^2} \sum_{j,m \in \mathbb{Z}_+} \widehat{f}_{jm} P_{z,mj} \cdot q^{-2j} \right) \overset{\circ}{\psi} \, d\nu(z). \qquad \square$$

Note that the integral representation (2.2) is a q-analogue of the relation (3.4) in [1].

On can deduce from the covariance of algebras $D(U)_q$, $F_{q,\alpha}$, the invariance of the integrals $\nu : D(U)_q \to \mathbb{C}$, $\mathrm{tr}_q : F_{q,\alpha} \to \mathbb{C}$, the "integration in parts" formula [9, proposition 2.1], and proposition 1.2 the following

Proposition 2.2 *The linear map $F_{q,\alpha} \to D(U)'_q$, $\widehat{f} \mapsto f$, which takes a linear operator to its covariant symbol, is a morphism of $U_q\mathfrak{sl}_2$-modules.*

As in [13], we call the covariant symbol f for the Toeplitz-Bergman operator $\widehat{f}$ with symbol $\overset{\circ}{f} \in D(U)_q$ a Berezin transform of the function $\overset{\circ}{f}$. The associated transform map will be denoted by $B_{q,\alpha}$:

$$B_{q,\alpha} : D(U)_q \to D(U)'_q; \qquad B_{q,\alpha} : \overset{\circ}{f} \mapsto f.$$

Propositions 1.2 and 2.2 imply

Proposition 2.3 *The Berezin transform is a morphism of $U_q\mathfrak{sl}_2$-modules.*

Example 2.4. Let $\widehat{f} \in F_{q,\alpha}$ be given by $\widehat{f} z^j = \begin{cases} 1, j = 0 \\ 0, j \neq 0 \end{cases}$. Then one has $f = (1 - zz^*)^{2\alpha+1}$. Hence, $B_{q,\alpha} f_0 = (1 - q^{4\alpha})(1 - zz^*)^{2\alpha+1}$ since $\widehat{f_0} : z^j \mapsto \begin{cases} 1 - q^{4\alpha} & , \quad j = 0 \\ 0 & , \quad j \neq 0 \end{cases}$ (see Example 1.3).

To conclude, we prove that Berezin transform is an integral operator, and find its kernel. In this way, a q-analogue of the relation (4.8) from [1] is to be obtained.

Proposition 2.5 *For all* $\overset{\circ}{f} \in D(U)_q$,

$$(B_{q,\alpha} \overset{\circ}{f})(z) = \int\limits_{U_q} b_{q,\alpha}(z, \zeta) \overset{\circ}{f} (\zeta) d\nu(\zeta),$$

with $b_{q,\alpha} \in D(U \times U)'_q$ *being given by*

$$b_{q,\alpha}(z, \zeta) = \frac{1 - q^{4\alpha}}{1 - q^2} (1 - zz^*)^{2\alpha+1} (1 - \zeta\zeta^*)^{2\alpha+1} \{(q^2 z^* \zeta; q^2)_{-(2\alpha+1)} \cdot (z\zeta^*; q^2)_{-(2\alpha+1)}\}.$$

(See [10] for the definition of $\{.,.\}$*.)*

Proof. Consider the linear operator

$$\widetilde{B}_{q,\alpha} : D(U)_q \to D(U)'_q; \qquad \widetilde{B}_{q,\alpha} : \overset{\circ}{f} \mapsto \int\limits_{U_q} b_{q,\alpha}(z, \zeta) \overset{\circ}{f} (\zeta) d\nu(\zeta).$$

Its kernel coincides up to a constant multiple to the invariant kernel $k_{22}^{-(2\alpha+1)} \cdot k_{11}^{-(2\alpha+1)}$ (see [10]). Hence, $\widetilde{B}_{q,\alpha}$ is a morphism of $U_q\mathfrak{sl}_2$-modules by [9, proposition 4.5]. Note that $B_{q,\alpha}$ possesses the same property. It was shown in [9] that $f_0 \in D(U)_q$ generates the $U_q\mathfrak{sl}_2$-module $D(U)_q$. In this context, the desired equality $\widetilde{B}_{q,\alpha} = B_{q,\alpha}$ becomes a consequence of $\widetilde{B}_{q,\alpha} f_0 = (1 - q^{4\alpha})(1 - zz^*)^{2\alpha+1} = B_{q,\alpha} f_0$. □

3 Berezin transform and Laplace-Beltrami operator

The following lemma is deduced from the relation

$$(1 - zz^*)^\lambda = \sum_{n=0}^{\infty} q^{2n\lambda} f_n, \qquad \lambda \in \mathbb{C}$$

and (1.3):

Lemma 3.1 *For all* $m, j \in \mathbb{Z}_+$ *the following decomposition is valid in* $D(U)'_q$*:*

$$P_{z,mj} = \frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m} \sum_{n=0}^{\infty} q^{4\alpha n} \cdot P_{z,mj}^{(n)}, \tag{3.1}$$

with $P_{z,mj}^{(n)} = q^{2(j+n)} z^j \cdot f_n \cdot z^{*m} \in D(U)_q$.

Let $\overset{\circ}{f}, \psi \in D(U)_q$. Consider the integral $\int\limits_{U_q} \psi^* \cdot B_{q,\alpha} \overset{\circ}{f} d\nu$ as a function of $t = q^{4\alpha}$. Now proposition 2.1 and lemma 3.1 imply the analyticity of this function as $t \in [0,1)$. Hence, one has

Proposition 3.2 *There exists a unique sequence of $U_q\mathfrak{sl}_2$-module morphisms $B_q^{(n)} : D(U)_q \to D(U)'_q$, $n \in \mathbb{Z}_+$, such that for all $\overset{\circ}{f} \in D(U)_q$*

$$B_{q,\alpha} \overset{\circ}{f} = \sum_{n=0}^{\infty} q^{4\alpha n} B_q^{(n)} \overset{\circ}{f} . \tag{3.2}$$

Our purpose is to prove that the linear operators $B_q^{(n)}$ are polynomials of Laplace-Beltrami operator in the quantum disc.

Let

$$p_j(t) = \sum_{k=0}^{j} \frac{(q^{-2j};q^2)_k}{(q^2;q^2)_k^2} q^{2k} \cdot \prod_{i=0}^{k-1} \left(1 - q^{2i}\left((1-q^2)^2 t + 1 + q^2\right) + q^{4i+2}\right). \tag{3.3}$$

Lemma 3.3 *$p_j(\Box) f_0 = q^{2j} \cdot f_j$ for all $j \in \mathbb{Z}_+$.*

Proof. Remind [8] that for all $l \in \mathbb{C}$ the basic hypergeometric series

$$\varphi_l = {}_3\Phi_2 \left[\begin{matrix} (1-zz^*)^{-1}, q^{-2l}, q^{2(l+1)}; q^2; q^2 \\ q^2, 0 \end{matrix} \right]$$

converge in $D(U)'_q$, and

$$\Box \varphi_l = -\frac{(1-q^{-2l})(1-q^{2l+2})}{(1-q^2)^2} \varphi_l.$$

By a virtue of [8, §6], it suffices to show that for all $l \in \mathbb{C}$

$$q^{-2j} \cdot \int\limits_{U_q} \varphi_l^* \cdot p_j(\Box) f_0 d\nu = \int\limits_{U_q} \varphi_l^* f_j d\nu. \tag{3.4}$$

After substituting l by $\bar{l}$ we find out that (3.4) is equivalent to

$$p_j \left(-\frac{(1-q^{-2l})(1-q^{2(l+1)})}{(1-q^2)^2} \right) = {}_3\Phi_2 \left[\begin{matrix} q^{-2j}, q^{-2l}, q^{2(l+1)}; q^2; q^2 \\ q^2, 0 \end{matrix} \right].$$

Prove this relation. By the definition of ${}_3\Phi_2$ one has

$${}_3\Phi_2 \left[\begin{matrix} q^{-2j}, q^{-2l}, q^{2(l+1)}; q^2; q^2 \\ q^2, 0 \end{matrix} \right] =$$

$$= \sum_{k=0}^{j} \frac{(q^{-2j};q^2)_k}{(q^2;q^2)_k^2} \cdot \prod_{i=0}^{k-1} \left((1 - q^{2i} \cdot q^{2(l+1)})(1 - q^{2i} \cdot q^{-2l}) \right) \cdot q^{2k} =$$

$$= \sum_{k=0}^{j} \frac{(q^{-2j};q^2)_k}{(q^2;q^2)_k^2} \cdot \prod_{i=0}^{k-1} \left((1+q^{2i}\cdot u + q^{4i+2}\right)\cdot q^{2k}$$

with $u = -q^{2l+2} - q^{-2l}$. It remains to prove that

$$p_j\left(-\frac{1+q^2+u}{(1-q^2)^2}\right) = \sum_{k=0}^{j} \frac{(q^{-2j};q^2)_k}{(q^2;q^2)_k^2} \cdot \prod_{i=0}^{k-1} \left((1+q^{2i}\cdot u + q^{4i+2}\right)\cdot q^{2k}.$$

For that, it suffices to exclude u by a substitution $u = -(1-q^2)^2 t - 1 - q^2$. □

The next statement refines essentially proposition 3.2.

Proposition 3.4 *For all* $\overset{\circ}{f}\in D(U)_q$ *the following expansion in* $D(U)'_q$ *is valid:*

$$B_{q,\alpha}\overset{\circ}{f} = (1-q^{4\alpha})\sum_{j\in\mathbb{Z}_+} q^{4\alpha\cdot j}\cdot p_j(\Box)\overset{\circ}{f}\ . \tag{3.5}$$

Proof. One has the relation $B_{q,\alpha}f_0 = (1-q^{4\alpha})\sum\limits_{k\in\mathbb{Z}_+} q^{(4\alpha+2)k} f_k$ (see example 2.4). Hence, in the special case $\overset{\circ}{f} = f_0$ our statement follows from lemma 3.3. It remains to take into account that f_0 generates the $U_q\mathfrak{sl}_2$-module $D(U)_q$, and the operators $B_{q,\alpha}$, $\Box$ are morphisms of $U_q\mathfrak{sl}_2$-modules (see [8, proposition 2.1]). □

Corollary 3.5

$$B_q^{(n)} = \begin{cases} I & ,\quad n=0 \\ p_n(\Box) - p_{n-1}(\Box) & ,\quad n\in\mathbb{N} \end{cases} \tag{3.6}$$

4 Toeplitz-Bergman operators with bounded symbols

It is very well known [5, 6] that the *-algebra $\mathrm{Pol}(\mathbb{C})_q$ has a unique up to unitary equivalence faithful irreducible representation. As it was described in [8], this representation T lives in a Hilbert space $\overline{H}$ constructed as a completion of the pre-Hilbert space H. Let $L(\overline{H})$ be the algebra of all bounded operators in H and H' the vector space of all bounded antilinear functionals on H. One has

$$\mathrm{End}_{\mathbb{C}}(H) \subset L(\overline{H}) \subset \mathrm{Hom}_{\mathbb{C}}(H, H').$$

It was demonstrated in [8] that the map $T : \mathrm{Pol}(\mathbb{C})_q \hookrightarrow L(\overline{H})$ is extendable by a continuity up to the isomorphism $T : D(U)'_q \overset{\rightarrow}{\approx} \mathrm{Hom}_{\mathbb{C}}(H, H')$.

We call a distribution $f \in D(U)'_q$ bounded if $T(f) \in L(\overline{H})$. Impose the notation

$$L_q^\infty = \{f\in D(U)'_q |\, T(f)\in L(\overline{H})\}, \qquad \|f\|_\infty = \|T(f)\|.$$

(It is easy to show that the algebra L^∞ defined in this way is isomorphic to the enveloping von Neumann algebra of the C^*-algebra of continuous functions in the quantum disk, which was considered, in particular, in [6]).

Consider the subspaces

$$\mathbb{C}[z]_{q,\infty}=\{f\in D(U)_q|\ f\cdot z=0\},$$

$$H^2_{q,\infty}=\{f\in L^2(U)_q|\ f\cdot z=0\},$$

$$\mathbb{C}[[z]]_{q,\infty}=\{f\in D(U)'_q|\ f\cdot z=0\}.$$

It follows from [9, proposition 3.3] that

$$\mathbb{C}[z]_{q,\infty}=\mathbb{C}[z]\cdot f_0,\qquad \mathbb{C}[[z]]_{q,\infty}=\mathbb{C}[[z]]\cdot f_0,$$

and hence

$$H\simeq\mathbb{C}[z]_{q,\infty},\qquad \overline{H}\simeq H^2_{q,\infty},\qquad H'\simeq\mathbb{C}[[z]]_{q,\infty}.$$

T is unitarily equivalent to the representation $\widehat{T}$ of $\mathrm{Pol}(\mathbb{C})_q$ in $H^2_{q,\infty}$ given by

$$\widehat{T}:\psi\mapsto f\cdot\psi;\qquad f\in\mathrm{Pol}(\mathbb{C})_q,\ \psi\in H^2_{q,\infty}\subset D(U)'_q.$$

Thus, a distribution $f\in D(U)'_q$ is bounded iff the linear operator $\widehat{T}(f)$ is in $L(H^2_{q,\infty})$; in this case $\|f\|_\infty=\|\widehat{T}(f)\|_\infty$.

The following proposition justifies the use of the symbol ∞ in the notation for the vector spaces $\mathbb{C}[z]_{q,\infty}$, $H^2_{q,\infty}$, $\mathbb{C}[[z]]_{q,\infty}$.

Proposition 4.1 *For any polynomial $\psi\in\mathbb{C}[z]_q$*

$$\lim_{\alpha\to\infty}(\psi,\psi)_{q,\alpha}=\frac{1}{1-q^2}(\psi f_0,\psi f_0).$$

Proof.

$$(\psi,\psi)_{q,\infty}\overset{\mathrm{def}}{=}\lim_{\alpha\to\infty}\frac{1-q^2}{1-q^{4\alpha}}(\psi,\psi)_{q,\alpha}=\lim_{\alpha\to\infty}\int\limits_{U_q}\psi^*\psi\sum_{n=0}^{\infty}q^{4n\alpha}f_n d\nu=$$

$$=\int\limits_{U_q}\psi^*\psi f_0 d\nu=(\psi f_0,\psi f_0).\qquad\Box$$

The following remark will not be used in the sequel. Proposition 4.1 allows one to prove that the covariant algebra $D(U)_q$ is isomorphic to a "limit $F_{q,\infty}$ of covariant algebras $F_{q,\alpha}$ as $\alpha\to\infty$". This leads to an alternate scheme of producing the covariant algebra $D(U)_q$ of finite functions in the quantum disk. Under this scheme, at the first step a unitarizable Harish-Chandra module V_α with lowest weight $\alpha>0$ and the covariant algebras $V_\alpha\otimes V_\alpha^*\hookrightarrow\mathrm{End}_{\mathbb{C}}(V_\alpha)$ are constructed. The second step is in "passage to the limit" $\lim\limits_{\alpha\to+\infty}V_\alpha\otimes V_\alpha^*$ which is to be declared the algebra of finite functions in the quantum disk.

Finally, impose the notation

$$\overline{F}_{q,\infty}\overset{\mathrm{def}}{=}\mathrm{End}_{\mathbb{C}}(\mathbb{C}[z]_{q,\infty},\mathbb{C}[[z]]_{q,\infty}).$$

It follows from the definitions that the representation $\widehat{T}$ is extendable up to a bijection $\widehat{T}:D(U)'\overset{\sim}{\to}\overline{F}_{q,\infty}$.

It should be noted that $\mathrm{Pol}(\mathbb{C})_q \subset L_q^\infty$. This can be deduced, for example, from the fact that the representation $\widehat{T}$ of $\mathrm{Pol}(\mathbb{C})_q$ in the pre-Hilbert space $\mathbb{C}[z]_{q,\infty}$ is a *-representation of this algebra. Hence, $I - \widehat{T}(z)\widehat{T}(z^*) \geq 0$, $\|\widehat{T}(z)\| = \|\widehat{T}(z^*)\| = 1$.

Let A be a compact linear operator in a Hilbert space and $|A| \stackrel{\mathrm{def}}{=} (A^*A)^{1/2}$. Consider the sequence of eigenvalues of $|A|$, with their multiplicities being taken into account:

$$s_1(A) \geq s_2(A) \geq \ldots .$$

The numbers $s_p(A)$, $p \in \mathbb{N}$, are called s-values of A.

Remind the notation S_∞ for the ideal of all compact operators in a Hilbert space, together with the notation

$$\|A\|_p = \left(\sum_{n\in\mathbb{N}} s_n(A)^p\right)^{1/p}, \qquad S_p = \{A \in S_\infty |\ \|A\|_p < \infty\}, \qquad p > 0,$$

for the normed ideals of von Neumann-Schatten (see [3]).

Lemma 4.2 *For any function $\psi \in D(U)_q$*

$$\|\psi\| = (1-q^2)^{1/2} \cdot \|\widehat{T}(\psi(1-zz^*)^{-1/2})\|_2,$$

with $\|\psi\| = \left(\int\limits_{U_q} \psi^*\psi d\nu\right)^{1/2}$.

Proof. It follows from (1.1) and the well known tracial properties of an operator $A \in S_1$ that

$$\|\psi\|^2 = (1-q^2)\mathrm{tr}\,\widehat{T}(\psi^*\psi(1-zz^*)^{-1}) = (1-q^2)\mathrm{tr}\,\widehat{T}((1-zz^*)^{-1/2}\psi^*\psi(1-zz^*)^{-1/2}). \quad \square$$

Corollary 4.3 *Let $\overset{\circ}{f} \in L_q^\infty$, $\psi \in D(U)_q$, then $\overset{\circ}{f}\psi \in L^2(d\nu)_q$ and $\|\overset{\circ}{f}\psi\| \leq \|\overset{\circ}{f}\|_\infty \cdot \|\psi\|$.*

Proof.

$$\|\overset{\circ}{f}\psi\| = (1-q^2)^{1/2}\|\widehat{T}(\overset{\circ}{f})\widehat{T}(\psi(1-zz^*)^{-1/2})\|_2 \leq$$
$$\leq (1-q^2)^{1/2}\|\widehat{T}(\overset{\circ}{f})\| \cdot \|\widehat{T}(\psi(1-zz^*)^{-1/2})\|_2 = \|\overset{\circ}{f}\|_\infty \cdot \|\psi\|. \quad \square$$

It follows from the boundedness of the multiplication operator by a bounded function $\overset{\circ}{f}$:

$$D(U)_q \to L^2(d\nu)_q, \qquad \psi \mapsto \overset{\circ}{f}\psi$$

that it admits an extension by a continuity onto the entire space $L^2(d\nu)_q$. This allows one to define a Toeplitz-Bergman operator $\widehat{f}$ with symbol $\overset{\circ}{f} \in L_q^\infty$:

$$\widehat{f} : H^2_{q,\alpha} \to H^2_{q,\alpha}; \qquad \widehat{f} : \psi \mapsto P_{q,\alpha}(\overset{\circ}{f}\psi).$$

By a virtue of corollary 4.3 one has

$$\|\widehat{f}\| \leq \| \overset{\circ}{f} \|_\infty, \tag{4.1}$$

with $\|\widehat{f}\|$ being the norm of the operator $\widehat{f}$ in $H^2_{q,\alpha}$. Thus we get a norm decreasing linear map $L^\infty_q \to L(H^2_{q,\alpha})$, $\overset{\circ}{f} \mapsto \widehat{f}$. This definition generalizes that of a Toeplitz-Bergman operator with finite symbol (see section 2).

5 Berezin transform: bounded functions

The $U_q\mathfrak{sl}_2$-module $\mathbb{C}[z]_{q,\alpha}$ is formed by polynomials $\psi = \sum\limits_{i \in \mathbb{Z}_+} a_i(\psi) z^i$. Consider a completion $\mathbb{C}[[z]]_{q,\alpha}$ of this vector space in the topology of coefficientwise convergence, and impose the notation $\overline{F}_{q,\alpha} \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[z]_{q,\alpha}, \mathbb{C}[[z]]_{q,\alpha})$ for the corresponding completion of $F_{q,\alpha}$. Equip $\overline{F}_{q,\alpha}$ with the topology of pointwise (strong) convergence:

$$\lim_{n\to\infty} A_n = A \quad \Leftrightarrow \quad \forall \psi \in \mathbb{C}[z]_{q,\alpha} \ \lim_{n\to\infty} A_n\psi = A\psi.$$

Evidently, $\mathbb{C}[z]_{q,\alpha} \subset H^2_{q,\alpha} \subset \mathbb{C}[[z]]_{q,\alpha}$, and so

$$F_{q,\alpha} \subset L(H^2_{q,\alpha}) \subset \overline{F}_{q,\alpha}.$$

The representation operators of E, F, $K^{\pm 1}$ in $\mathbb{C}[z]_{q,\alpha}$ have degrees $+1$, -1, 0 respectively. Hence they are extendable by a continuity from $\mathbb{C}[z]_{q,\alpha}$ onto $\mathbb{C}[[z]]_{q,\alpha}$, and from $F_{q,\alpha}$ onto $\overline{F}_{q,\alpha}$.

Of course, $\overline{F}_{q,\alpha}$ is a covariant bimodule over the covariant algebra $F_{q,\alpha}$. It is easy to show that the linear functional

$$F_{q,\alpha} \otimes F_{q,\alpha} \to \mathbb{C}, \qquad \widehat{f} \otimes \widehat{\psi} \mapsto \mathrm{tr}_q(\widehat{f}\widehat{\psi})$$

is extendable by a continuity up to a morphism of $U_q\mathfrak{sl}_2$-modules $\overline{F}_{q,\alpha} \otimes F_{q,\alpha} \to \mathbb{C}$.

Define a covariant symbol $f \in D(U)'_q$ of a linear operator $\widehat{f} \in \overline{F}_{q,\alpha}$ by (2.1). The map $\overline{F}_{q,\alpha} \to D(U)'_q$ arising this way is a $U_q\mathfrak{sl}_2$-module morphism.

In the following proposition we use notation $\widehat{f}$ for a linear operator without assuming it to be a Toeplitz-Bergman operator.

Proposition 5.1 *Let $\widehat{f}$ be a linear operator*

$$\widehat{f} : \mathbb{C}[z]_{q,\alpha} \to \mathbb{C}[[z]]_{q,\alpha}, \qquad \widehat{f} : z^j \mapsto \sum_{m \in \mathbb{Z}_+} \widehat{f}_{mj} z^m, \quad j \in \mathbb{Z}_+.$$

The series $\sum\limits_{j,m \in \mathbb{Z}_+} \widehat{f}_{jm} P_{z,mj} q^{-2j}$ *converges in $D(U)'_q$ to the covariant symbol of $\widehat{f}$.*

Proof. It follows from the results of section 1 that for any $\overset{\circ}{\psi} \in D(U)_q$ all but finitely many of integrals $\int\limits_{U_q} P_{z,mj} \overset{\circ}{\psi}(z) d\nu(z)$ are zero. This allows one to reproduce literally the argument used in the proof of proposition 2.1. □

Let $\overset{\circ}{f} \in L_q^\infty$, and $\widehat{f} \in L(H^2_{q,\alpha}) \subset \overline{F}_{q,\alpha}$ be the Toeplitz-Bergman operator with symbol $\overset{\circ}{f}$. We follow [13] in using the term "Berezin transform of the function $\overset{\circ}{f}$" for the covariant symbol of the linear operator $\widehat{f}$.

Our purpose is to decompose the operator-function $B_{q,\alpha} : L_q^\infty \to D(U)'_q$ into series in powers of $t = q^{4\alpha}$ (cf. (3.5)).

One can use again the argument of proposition 1.5 to get (1.4) for all bounded symbols $\overset{\circ}{f} \in L_q^\infty$. An application of (1.1) and the fact that $\widehat{T}((1-zz^*)^{2\alpha})$ is a trace class operator for all $\alpha > 0$, yields also

Proposition 5.2 *For all $\overset{\circ}{f} \in L_q^\infty$ $m, j \in \mathbb{Z}_+$*

$$\widehat{f}_{mj} = \frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m} \cdot (1 - q^{4\alpha}) \cdot \mathrm{tr}\left(\widehat{T}(z^j(1-zz^*)^{2\alpha} z^{*m})\widehat{T}(\overset{\circ}{f})\right).$$

Let Θ be the vector space of holomorphic functions in the unit disc with values in the Banach algebra S_1 of trace class operators in $\overline{H}$. (Each function $Q(t)$ from Θ admits a expansion into the power series $Q(t) = \sum\limits_{n \in \mathbb{Z}_+} t^n \cdot Q^{(n)}$ with $\lim\limits_{n\to\infty} \|Q^{(n)}\|_1^{1/n} \leq 1$.)

Proposition 5.3 *For all $j, m \in \mathbb{Z}_+$*

$$\sum_{n \in \mathbb{Z}_+} t^n \cdot \widehat{T}(z^j \cdot f_n \cdot z^{*m}) \in \Theta.$$

Proof. Remind that $\widehat{T}(z)\widehat{T}(z^*) = 1 - \sum\limits_{n \in \mathbb{Z}_+} q^{2n} \cdot \widehat{T}(f_n)$, and that $\widehat{T}(f_n)$ are one-dimensional projections, $n \in \mathbb{Z}_+$. Hence $\|\widehat{T}(z)\| = \|\widehat{T}(z^*)\| = \|\widehat{T}(f_n)\|_1 = 1$. Finally,

$$\|\widehat{T}(z^j f_n z^{*m})\|_1 \leq \|\widehat{T}(z)\|^j \cdot \|\widehat{T}(f_n)\|_1 \cdot \|\widehat{T}(z^*)\|^m = 1. \qquad \square$$

Propositions 5.2, 5.3 and the definition of Berezin transform imply

Corollary 5.4 *Let $\psi \in D(U)_q$. There exists a unique function $Q_\psi(t) \in \Theta$ such that*

$$\int_{U_q} (B_{q,\alpha} \overset{\circ}{f})\psi d\nu = \mathrm{tr}\left(\widehat{T}(\overset{\circ}{f}) Q_\psi(q^{4\alpha})\right) \tag{5.1}$$

for all $\overset{\circ}{f} \in L_q^\infty$.

Proof. The uniqueness of $Q_\psi(t)$ is evident. In fact, given such $A \in L(\overline{H})$ that for all $\overset{\circ}{f} \in D(U)_q$ one has $\mathrm{tr}\left(\widehat{T}(\overset{\circ}{f})A\right) = 0$, then surely $A = 0$. The existence of $Q_\psi \in \Theta$ follows from propositions 5.2, 5.3 and the definition of Berezin transform. $\square$

The coefficients of the Taylor series for the holomorphic function $Q_\psi(t)$ at $t = 0$ are trace class operators. One can use (3.5) and (1.1) to express those coefficients via the operators $\widehat{T}(p_j(\Box)\psi)$, $j \in \mathbb{Z}_+$. Thus we get the following

Proposition 5.5 *Let* $\overset{\circ}{f} \in L_q^\infty$.

1. *For all* $\alpha > 0$ *one has a expansion in* $D(U)'_q$
$$B_{q,\alpha}\overset{\circ}{f} = \sum_{n\in\mathbb{Z}_+} q^{4\alpha n} B_q^{(n)} \overset{\circ}{f}\ .$$

2. *For all* $\psi \in D(U)_q$ *one has the asymptotic expansion*
$$\int\limits_{U_q}\left(B_{q,\alpha}\overset{\circ}{f}\right)\psi d\nu \underset{\alpha\to+\infty}{\sim} \sum_{n=0}^{\infty} q^{4\alpha n}\int\limits_{U_q}\left(B_q^{(n)}\overset{\circ}{f}\right)\psi d\nu. \tag{5.2}$$

Here $B_q^{(n)} : D(U)'_q \to D(U)'_q$ *are polynomial functions of the Laplace-Beltrami operator, given explicitly by (3.6).*

6 Covariant symbols

The notation $\widehat{z}$, $\widehat{z}^*$ in [8] stand for the Toeplitz-Bergman operators with symbols z, z^*. Those are defined in the graded vector space $\mathbb{C}[z]_{q,\alpha}$, with $\deg(\widehat{z}) = +1$, $\deg(\widehat{z}^*) = -1$. Hence for any matrix $(a_{ij})_{i,j\in\mathbb{Z}_+}$ with numerical entries, series
$$\widehat{f} = \sum_{i,j\in\mathbb{Z}_+} a_{ij}\widehat{z}^i\widehat{z}^{*j} \tag{6.1}$$
converge in the topological vector space $\overline{F}_{q,\alpha} = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[z]_{q,\alpha}, \mathbb{C}[[z]]_{q,\alpha})$.

Proposition 6.1 $\widehat{f}z^n = \sum\limits_{m\in\mathbb{Z}_+} b_{mn}z^m$, $n \in \mathbb{Z}_+$,

with $b_{mn} = \sum\limits_{j=0}^{\min(m,n)} \dfrac{(q^{2n};q^{-2})_{n-j}}{(q^{4\alpha+2n};q^{-2})_{n-j}} a_{m-j,n-j}$.

Proof. It suffices to apply the relations
$$\widehat{z}(z^m) = z^{m+1}, \qquad \widehat{z}^*(z^m) = \begin{cases} \dfrac{1-q^{2m}}{1-q^{4\alpha+2m}}\cdot z^{m-1} & , \quad m \neq 0 \\ 0 & , \quad m = 0 \end{cases} \tag{6.2}$$
which were established in [8, section 7] (see also [5]). □

Corollary 6.2 *For any linear operator* $\widehat{f} \in \overline{F}_{q,\alpha}$ *there exists a unique decomposition (6.1).*

Example 6.3. Consider the linear operator $\widehat{f}_0 : z^j \mapsto \begin{cases} 1-q^{4\alpha} & , \quad j=0 \\ 0 & , \quad j \neq 0 \end{cases}, \; j \in \mathbb{Z}_+$. Prove that

$$\widehat{f}_0 = (1-q^{4\alpha}) \sum_{k=0}^{\infty} \frac{(q^{-4\alpha-2}; q^2)_k}{(q^2; q^2)_k} q^{(4\alpha+2)k} \widehat{z}^k \widehat{z}^{*k}. \tag{6.3}$$

Pass from the equality of operators to the equalities of their matricial elements with respect to the base $\{z^n\}_{n \in \mathbb{Z}_+}$. Of course, all the non-diagonal elements are zero. An identification of the diagonal elements yields

$$\sum_{k=0}^{j} \frac{(q^{-4\alpha-2}; q^2)_k}{(q^2; q^2)_k} \cdot \frac{(q^{2j}; q^{-2})_k}{(q^{4\alpha+2j}; q^{-2})_k} \cdot q^{(4\alpha+2)k} = \delta_{j0}. \tag{6.4}$$

It suffices to consider the case $j > 0$. Multiply (6.4) by $\dfrac{(q^{4\alpha+2j}; q^{-2})_j}{(q^{2j}; q^{-2})_j}$ to get

$$\sum_{k=0}^{j} \frac{(q^{-4\alpha-2}; q^2)_k}{(q^2; q^2)_k} \cdot \frac{(q^{4\alpha+2}; q^2)_{j-k}}{(q^2; q^2)_{j-k}} \cdot q^{(4\alpha+2)k} = 0.$$

That is,

$$\sum_{k+m=j} \frac{(q^{-4\alpha-2}; q^2)_k}{(q^2; q^2)_k} \cdot q^{(4\alpha+2)k} \cdot \frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m} = 0.$$

So, it remains to consider the q-binomial series (see [4]):

$$a(t) = \sum_{k \in \mathbb{Z}_+} \frac{(q^{-4\alpha-2}; q^2)_k}{(q^2; q^2)_k} \cdot q^{(4\alpha+2)k} \cdot t^k = \frac{(t; q^2)_\infty}{(q^{4\alpha+2}t; q^2)_\infty},$$

$$b(t) = \sum_{m \in \mathbb{Z}_+} \frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m} \cdot t^m = \frac{(q^{4\alpha+2}t; q^2)_\infty}{(t; q^2)_\infty},$$

and to observe that $a(t) \cdot b(t) = 1$. □

It was noted in section 1 that $\widehat{f}_0$ is a Toeplitz-Bergman operator with symbol f_0. This element generates the topological $U_q\mathfrak{sl}_2$-module $\overline{F}_{q,\alpha}$, as one can see from

Proposition 6.4 $U_q\mathfrak{sl}_2 \widehat{f}_0 = F_{q,\alpha}$.

Proof. Since for all $i, j, n \in \mathbb{Z}_+$, $\widehat{z}^i \widehat{f}_0 \widehat{z}^{*n} : z^j \mapsto (1-q^{4\alpha}) \cdot \dfrac{(q^2; q^2)_n}{(q^{4\alpha+2}; q^2)_n} \cdot \delta_{jn} z^i$, the linear operators $\{\widehat{z}^i \widehat{f}_0 \widehat{z}^{*n}\}_{i,n \in \mathbb{Z}_+}$ generate $F_{q,\alpha}$ as a vector space. It remains to show that all those operators are in the $U_q\mathfrak{sl}_2$-module generated by $\widehat{f}_0$. For that, it suffices to reproduce the proof of [9, theorem 3.9]. One has only to alter the notation for the generators (now they are $\widehat{z}$, $\widehat{z}^*$, $\widehat{f}_0$), together with the constants in formulae which describe the action of $X^{\pm}$ on $\widehat{f}_0$:

$$X^+ \widehat{f}_0 = c' \widehat{z} \cdot \widehat{f}_0; \qquad X^- \widehat{f}_0 = c'' \widehat{f}_0 \cdot \widehat{z}^*; \qquad c', c'' \neq 0.$$

These relations follow from proposition 1.2, [9, proposition 3.8], and

$$\mathbb{C}\widehat{z f_0} = \mathbb{C}\widehat{z}\widehat{f}_0, \qquad \mathbb{C}\widehat{f_0 z^*} = \mathbb{C}\widehat{f}_0\widehat{z^*}.$$

The latter relations can be deduced from

$$\operatorname{Im}\widehat{zf_0} = \operatorname{Im}\widehat{f_0 z^*} = \mathbb{C}; \qquad \operatorname{Ker}\widehat{f_0} = \operatorname{Ker}\widehat{zf_0} = \mathbb{C}^{\perp}.$$

It was shown in [8, section 1] that for any $f \in D(U)'_q$ there exists a unique decomposition $f = \sum\limits_{j,n\in\mathbb{Z}_+} a_{jk} z^j z^{*k}$ similar to (6.1).

EXAMPLE 6.5. Prove that

$$(1-q^{4\alpha})(1-zz^*)^{2\alpha+1} = (1-q^{4\alpha}) \sum_{k\in\mathbb{Z}_+} \frac{(q^{-(4\alpha+2)};q^2)_k}{(q^2;q^2)_k} q^{(4\alpha+2)k} z^k z^{*k}. \tag{6.5}$$

Apply the operator $\widehat{T}$ to the both parts of (6.5) and identify the matricial elements with respect to the base $\{z^m\}$ (it suffices to consider the diagonal elements).

Use the relations [1]

$$\widehat{T}(z)z^m = z^{m+1}, \qquad \widehat{T}(z^*)z^m = \begin{cases} (1-q^{2m})z^{m-1} & , \quad m \neq 0 \\ 0 & , \quad m = 0 \end{cases}$$

to get

$$\sum_{k=0}^{j} \frac{(q^{-4\alpha-2};q^2)_k}{(q^2;q^2)_k} \cdot (q^{2j};q^{-2})_k \cdot q^{(4\alpha+2)k} = q^{2j(2\alpha+1)},$$

$$\sum_{k+m=j} \frac{(q^{-4\alpha-2};q^2)_k}{(q^2;q^2)_k} \cdot q^{(4\alpha+2)k} \cdot \frac{1}{(q^2;q^2)_m} = \frac{q^{2j(2\alpha+1)}}{(q^2;q^2)_j}.$$

It remains to pass to the q-binomial decompositions (see [4]) in the both sides of the obvious relation $a(t)b(t) = c(t)$, with

$$a(t) = \frac{(t;q^2)_\infty}{(q^{4\alpha+2}t;q^2)_\infty}; \qquad b(t) = \frac{1}{(t;q^2)_\infty}; \qquad c(t) = \frac{1}{(q^{4\alpha+2}t;q^2)_\infty}. \qquad \square$$

Proposition 6.6 *The covariant symbol of the operator* $\widehat{f} = \sum\limits_{j,k\in\mathbb{Z}_+} a_{jk} \widehat{z}^j \widehat{z}^{*k}$ *is* $f = \sum\limits_{j,k\in\mathbb{Z}_+} a_{jk} z^j z^{*k}$.

Proof. Let $S'_{q,\alpha} : \overline{F}_{q,\alpha} \to D(U)'_q$ be the map which takes a linear operator $\widehat{f} \in \overline{F}_{q,\alpha}$ to its covariant symbol. We have to prove that this map coincides with the map $S''_{q,\alpha} : \overline{F}_{q,\alpha} \to D(U)'_q$, given by $S''_{q,\alpha} : \sum\limits_{j,k\in\mathbb{Z}_+} a_{jk} \widehat{z}^j \widehat{z}^{*k} \mapsto \sum\limits_{j,k\in\mathbb{Z}_+} a_{jk} z^j z^{*k}$. The linear operators S', S'' are morphisms of $U_q\mathfrak{sl}_2$-modules, and the element $\widehat{f}_0$ generates the topological $U_q\mathfrak{sl}_2$-module $\overline{F}_{q,\alpha}$ by proposition 6.4. Thus it suffices to obtain the relation $S'\widehat{f}_0 = S''\widehat{f}_0$. It was shown in section 2 that $S'(\widehat{f}_0) = (1-q^{4\alpha})(1-zz^*)^{2\alpha+1}$. So it remains to see that $S''(\widehat{f}_0) = (1-q^{4\alpha})(1-zz^*)^{2\alpha+1}$. This follows from (6.3), (6.5). $\square$

[1] These relations can be deduced from (6.2) via passage to the limit as $\alpha \to \infty$,

Corollary 6.7 *The map $\overline{F}_{q,\alpha} \to D(U)'_q$ which takes a linear operator to its covariant symbol is one-to-one.*

To conclude, we give another illustration of corollary 6.2. Our immediate purpose is to get the expansion $\widehat{z}^*\widehat{z} = \sum\limits_{k \in \mathbb{Z}_+} c_k \widehat{z}^k \widehat{z}^{*k}$ and to find a generating function $c(u) = \sum\limits_{k \in \mathbb{Z}_+} c_k u^k$.

By (6.2), the coefficients c_k can be found from the system of equations

$$\sum_{k=0}^{m} c_k \frac{(q^{2m}; q^{-2})_k}{(q^{4\alpha+2m}; q^{-2})_k} = \frac{1 - q^{2(m+1)}}{1 - q^{4\alpha+2(m+1)}}, \qquad m \in \mathbb{Z}_+. \tag{6.6}$$

Apply an expansion of the right hand side of (6.6) as series:

$$\frac{1 - q^{2(m+1)}}{1 - q^{4\alpha+2(m+1)}} = 1 + \sum_{j \in \mathbb{N}} (1 - q^{-4\alpha}) q^{(2\alpha+1+m)2j}.$$

For a fixed $j \in \mathbb{N}$ consider the system of equations

$$\sum_{k=0}^{m} \gamma_k \frac{(q^{2m}; q^{-2})_k}{(q^{4\alpha+2m}; q^{-2})_k} = q^{2mj}, \qquad m \in \mathbb{Z}_+. \tag{6.7}$$

Multiply (6.7) by $\frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m}$ and convert it to the form

$$\sum_{i+k=m} \gamma_k \frac{(q^{4\alpha+2}; q^2)_i}{(q^2; q^2)_i} = q^{2mj} \frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m}. \tag{6.8}$$

Introduce the generating functions

$$\alpha(u) = \sum_{m \in \mathbb{Z}_+} \frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m} q^{2mj} u^m = \frac{(q^{4\alpha+2+2j}u; q^2)_\infty}{(q^{2j}u; q^2)_\infty},$$

$$\beta(u) = \sum_{i \in \mathbb{Z}_+} \frac{(q^{4\alpha+2}; q^2)_i}{(q^2; q^2)_i} u^i = \frac{(q^{4\alpha+2}u; q^2)_\infty}{(u; q^2)_\infty}.$$

It follows from(6.8) that

$$\gamma(u) \stackrel{\mathrm{def}}{=} \sum_{k \in \mathbb{Z}_+} \gamma_k u^k = \frac{\alpha(u)}{\beta(u)} = \frac{(u; q^2)_j}{(q^{4\alpha+2}u; q^2)_j}.$$

Turn back to the initial system (6.6) to obtain

$$c(u) = 1 + \sum_{j \in \mathbb{N}} (1 - q^{-4\alpha}) q^{(2\alpha+1)2j} \frac{(u; q^2)_j}{(q^{4\alpha+2}u; q^2)_j}. \tag{6.9}$$

7 $*$ - Product

Let A be an algebra over $\mathbb{C}$. Impose the notation

$$\mathbb{C}[[q^{4\alpha}]] = \left\{ \sum_{n\in\mathbb{Z}_+} q^{4\alpha} u_n |\, u_n \in \mathbb{C},\, n \in \mathbb{Z}_+ \right\},$$

$$A[[q^{4\alpha}]] = \left\{ \sum_{n\in\mathbb{Z}_+} q^{4\alpha} a_n |\, a_n \in A \right\}$$

for the ring of formal series with complex coefficients and the $\mathbb{C}[[q^{4\alpha}]]$-algebra of formal series with coefficients from A.

Our goal is to derive a new "distorted" multiplication in the $\mathbb{C}[[q^{4\alpha}]]$-algebra $\mathrm{Pol}(\mathbb{C})_q[[q^{4\alpha}]]$ from an ordinary multiplication in the $\mathbb{C}[[q^{4\alpha}]]$-algebra $\mathrm{End}(\mathbb{C}[z]_{q,\infty})[[q^{4\alpha}]]$.

The presence of the base $\{z^m\}_{m=0}^{\infty}$ in each vector space $\mathbb{C}[z]_{q,\alpha}$, $\mathbb{C}[z]_{q,\infty}$ allows one to "identify" them via the isomorphisms $i_\alpha : \mathbb{C}[z]_{q,\infty} \to \mathbb{C}[z]_{q,\alpha}$; $i_\alpha : z^m \mapsto z^m$, $m \in \mathbb{Z}_+$.

Consider the linear operators $i_\alpha^{-1} \widehat{z}^j \widehat{z}^{*k} i_\alpha$, $j,k \in \mathbb{Z}_+$ in $\mathbb{C}[z]_{q,\infty}$. It follows from (6.2) that

$$i_\alpha^{-1} \widehat{z}^j \widehat{z}^{*k} i_\alpha : z^m \mapsto \frac{(q^{2m}; q^{-2})_k}{(q^{4\alpha+2m}; q^{-2})_k} \cdot z^{m-k+j}, \quad m \in \mathbb{Z}_+. \tag{7.1}$$

From now on we shall identify the rational function $\frac{1}{(q^{4\alpha+2m}; q^{-2})_k}$ of an indeterminate $t = q^{4\alpha}$ with its q-binomial series (see [4])

$$\frac{(q^{4\alpha+2m+2}; q^2)_\infty}{(q^{4\alpha+2m+2-2k}; q^2)_\infty} = \sum_{n\in\mathbb{Z}_+} \left(\frac{(q^{2k}; q^2)_n}{(q^2; q^2)_n} \cdot q^{2(m-k+1)n} \right) q^{4\alpha n}.$$

The construction of $*$-product will be done via the $\mathbb{C}[[q^{4\alpha}]]$-linear map

$$Q : \mathrm{Pol}(\mathbb{C})_q[[q^{4\alpha}]] \to \mathrm{End}(\mathbb{C}[z]_{q,\infty})[[q^{4\alpha}]]$$

defined as follows:

$$Q : \sum_{n\in\mathbb{Z}_+} q^{4\alpha n} \sum_{j,k=1}^{N(n)} a_{jk}^{(n)} z^j z^{*k} \mapsto \sum_{n\in\mathbb{Z}_+} q^{4\alpha n} \sum_{j,k=1}^{N(n)} a_{jk}^{(n)} i_\alpha^{-1} (\widehat{z}^j \widehat{z}^{*k}) i_\alpha$$

for all numbers $a_{jk}^{(n)} \in \mathbb{C}$.

Lemma 7.1 *The map Q is injective.*

Proof. In the case Q has a non-trivial kernel, there should be for some $j,k \in \mathbb{Z}_+$, $\sum_{n\in\mathbb{Z}_+} c_n \widehat{z}^{j+n} \widehat{z}^{*(k+n)} = 0$, with $c_n \in \mathbb{C}[[q^{4\alpha}]]$, $n \in \mathbb{Z}_+$, and $c_0 \neq 0$. An application of the operator $\sum_{n\in\mathbb{Z}_+} c_n \widehat{z}^{j+n} \widehat{z}^{*(k+n)}$ to the vector z^k yields $c_0 \cdot \frac{(q^{2k}; q^{-2})_k}{(q^{4\alpha+2k}; q^{-2})_k} \cdot z^j = 0$, which is a contradiction. □

Lemma 7.2 *Let $j,k \in \mathbb{Z}_+$ and $\overset{\circ}{f} = z^{*j}z^k$. The Toeplitz-Bergman operator $\widehat{f}$ with symbol $\overset{\circ}{f}$ is $\widehat{z}^{*j}\widehat{z}^k$.*

Proof. For all $\psi_1, \psi_2 \in H^2_{q,\alpha}$ one has

$$(\widehat{f}\psi_1, \psi_2)_{q,\alpha} = (P_{q,\alpha}(z^{*j}z^k\psi_1), \psi_2)_{q,\alpha} = (z^{*j}z^k\psi_1, \psi_2)_{q,\alpha} = (z^k\psi_1, z^j\psi_2)_{q,\alpha} =$$
$$= (\widehat{z}^k\psi_1, \widehat{z}^j\psi_2)_{q,\alpha} = (\widehat{z}^{*j}\widehat{z}^k\psi_1, \psi_2)_{q,\alpha}. \qquad \square$$

The main result of this section is

Proposition 7.3 *There exists a unique $\mathbb{C}[[q^{4\alpha}]]$-bilinear map*

$$* : \mathrm{Pol}(\mathbb{C})_q[[q^{4\alpha}]] \times \mathrm{Pol}(\mathbb{C})_q[[q^{4\alpha}]] \to \mathrm{Pol}(\mathbb{C})_q[[q^{4\alpha}]]$$

such that $Q(f_1 * f_2) = (Qf_1)\cdot(Qf_2)$ for all $f_1, f_2 \in \mathrm{Pol}(\mathbb{C})_q[[q^{4\alpha}]]$.

Proof. The uniqueness follows from lemma 7.1. The existence of this $\mathbb{C}[[q^{4\alpha}]]$-bilinear map will be established via verifying an explicit formula (7.4). We start with considering the case $f_1 = z^*,\ f_2 = z$.

In section 6 a generating function $c(u) = \sum\limits_{k\in\mathbb{Z}_+} c_k u^k$ for the coefficients of the expansion

$$\widehat{z}^*\widehat{z} = \sum_{k\in\mathbb{Z}_+} c_k \widehat{z}^k \widehat{z}^{*k} \tag{7.2}$$

was derived. Prove that

$$B_{q,\alpha}(z^*z) = \sum_{k\in\mathbb{Z}_+} c_k z^k z^{*k}. \tag{7.3}$$

In fact, the distribution $B_{q,\alpha}(z^*z)$ coincides with the covariant symbol of the Toeplitz-Bergman operator with symbol z^*z. This operator is $\widehat{z}^*\widehat{z}$ by a virtue of corollary 7.2. Its covariant symbol is $\sum\limits_{k\in\mathbb{Z}_+} c_k z^k z^{*k}$ due to proposition 6.6.

It should be noted that $B_{q,\alpha}(z^*z) \in \mathrm{Pol}(\mathbb{C})_q[[q^{4\alpha}]]$. In fact, (6.9) implies

$$c(u) = c(u, q^{4\alpha}) = \sum_{n\in\mathbb{Z}_+} q^{4\alpha n}\cdot P_n(u),$$

with $P_n(u)$ being a polynomial of a degree at most $n+1$. Now our statement in the case $f_1 = z^*,\ f_2 = z$ follows from (7.2) and (7.3):

$$z^* * z = B_{q,\alpha}(z^*z).$$

In a more general setting $f_1 = z^{*m},\ f_2 = z^k,\ m,k \in \mathbb{Z}_+$, one can use a similar argument. One has:

$$z^{*m} * z^k = B_{q,\alpha}(z^{*m}z^k).$$

The relations $Q(zf) = zQ(f),\ Q(fz^*) = Q(f)z^*,\ f \in \mathrm{Pol}(\mathbb{C})_q[[q^{4\alpha}]]$, allow one to consider even more general case of $f_1, f_2 \in \mathrm{Pol}(\mathbb{C})_q$:

$$z^i z^{*m} * z^k z^{*j} = z^i B_{q,\alpha}(z^{*m}z^k) z^{*j}; \qquad i,j,k,m \in \mathbb{Z}_+. \tag{7.4}$$

To complete the proof of proposition 7.3, it remains to define the $*$-product of formal series:

$$\sum_{i\in\mathbb{Z}_+} q^{4\alpha i} f_1^{(i)} * \sum_{j\in\mathbb{Z}_+} q^{4\alpha j} f_2^{(j)} \stackrel{\mathrm{def}}{=} \sum_{n\in\mathbb{Z}_+} q^{4\alpha n} \left(\sum_{i+j=n} f_1^{(i)} * f_2^{(j)} \right),$$

with $f_1^{(i)}, f_2^{(j)} \in \mathrm{Pol}(\mathbb{C})_q$, $i, j \in \mathbb{Z}_+$. □

REMARK 7.4. The polynomials $P_n(u)$, $n \in \mathbb{Z}_+$, could be found without application of the *explicit formula* for generating function (6.9). In fact, if one sets up $c_k = \sum\limits_{n\in\mathbb{Z}_+} q^{4\alpha n} c_k^{(n)}$,

$$\widehat{z}^*\widehat{z} = \sum_{n\in\mathbb{Z}_+} q^{4\alpha n} \sum_{k=0}^{n+1} c_k^{(n)} \widehat{z}^k \widehat{z}^{*k}. \tag{7.5}$$

The constants $c_k^{(n)}$ could be found from the relation (see [8])

$$\widehat{z}^*\widehat{z} = q^2 \widehat{z}\widehat{z}^* + 1 - q^2 + q^{4\alpha} \cdot \frac{1-q^2}{1-q^{4\alpha}} \cdot (1 - \widehat{z}\widehat{z}^*)(1 - \widehat{z}^*\widehat{z}).$$

(For example, $P_0 = c_0^{(1)} u + c_0^{(0)} = q^2 u + 1 - q^2$.) This kind of description for coefficients in (7.5) was used in [8]. We observe that the $*$-product introduced here coincides with the $*$-product considered in [8].

8 $*$ - Product and q-differential operators

The operators $\square$, $\frac{\partial^{(l)}}{\partial z^*}$, $\frac{\partial^{(r)}}{\partial z^*}$, $\frac{\partial^{(l)}}{\partial z}$, $\frac{\partial^{(r)}}{\partial z}$ were introduced in [8].

Lemma 8.1 *Let φ, ψ be polynomials of one indeterminate. Then*

$$\frac{\partial^{(r)}}{\partial z^*}(\varphi(z^*)\psi(z)) = \frac{\partial^{(r)}\varphi(z^*)}{\partial z^*} \cdot \psi(q^2 z).$$

Proof. Since $dz^* \cdot z = q^2 z \cdot dz^*$, one has

$$\frac{\partial^{(r)}}{\partial z^*}(\varphi(z^*)\psi(z)) \cdot dz^* = \overline{\partial}(\varphi(z^*)\psi(z)) = (\overline{\partial}\varphi(z^*))\psi(z) =$$

$$= \frac{\partial^{(r)}\varphi(z^*)}{\partial z^*} \cdot dz^* \cdot \psi(z) = \frac{\partial^{(r)}\varphi(z^*)}{\partial z^*} \cdot \psi(q^2 z) \cdot dz^*. \qquad \square$$

Lemma 8.2 *For all $\psi(z) \in \mathbb{C}[z]_q$, $\frac{\partial^{(r)}\psi(z)}{\partial z} = \frac{\partial^{(l)}\psi(q^2 z)}{\partial z}$.*

Proof. Since $dz \cdot z = q^2 z \cdot dz$, one has

$$\frac{\partial^{(r)}\psi(z)}{\partial z} \cdot dz = \partial\psi = dz \cdot \frac{\partial^{(l)}\psi(z)}{\partial z} = \frac{\partial^{(l)}\psi(q^2 z)}{\partial z} \cdot dz. \qquad \square$$

Proposition 8.3 *Let f_1, f_2 be polynomials of one indeterminate. Then*

$$\Box(f_2(z^*)f_1(z)) = q^2 \cdot \frac{\partial^{(r)} f_2}{\partial z^*} \cdot (1 - zz^*)^2 \cdot \frac{\partial^{(l)} f_1}{\partial z}. \tag{8.1}$$

Proof. It follows from [11, corollary 2.9] that

$$\Box(f_2(z^*)f_1(z)) = q^2 \left(\frac{\partial^{(r)}}{\partial z^*} \frac{\partial^{(r)}}{\partial z}(f_2(z^*)f_1(z)) \right)(1 - zz^*)^2 =$$

$$= q^{-2} \left(\frac{\partial^{(r)}}{\partial z^*} \left(f_2(z^*) \frac{\partial^{(r)} f_1(z)}{\partial z} \right) \right)(1 - z^*z)^2.$$

Apply lemmas 8.1, 8.2 to conclude that

$$\Box(f_2(z^*)f_1(z)) = q^{-2} \frac{\partial^{(r)} f_2(z^*)}{\partial z^*} \cdot \frac{\partial^{(l)} f_1(q^4 z)}{\partial z}(1 - z^*z)^2.$$

It remains to apply the commutation relation $z(1 - z^*z)^2 = q^{-4}(1 - z^*z)^2 z$. $\Box$

Remind the notation from [8]:

$$\widetilde{\Box} = q^{-2}(1 - (1 + q^{-2})z^* \otimes z + q^{-2} z^{*2} \otimes z^2) \cdot \frac{\partial^{(r)}}{\partial z^*} \otimes \frac{\partial^{(l)}}{\partial z},$$

$m : \mathrm{Pol}(\mathbb{C})_q \otimes \mathrm{Pol}(\mathbb{C})_q \to \mathrm{Pol}(\mathbb{C})_q$, $m : \psi_1 \otimes \psi_2 \mapsto: \psi_1 \psi_2$.

Now we are in a position to prove [8, theorem 7.3].

Theorem 8.4 *For all $f_1, f_2 \in \mathrm{Pol}(\mathbb{C})_q$*

$$f_1 * f_2 = (1 - q^{4\alpha}) \cdot \sum_{j \in \mathbb{Z}_+} q^{4\alpha \cdot j} m(p_j(\widetilde{\Box}) f_1 \otimes f_2),$$

with p_j, $j \in \mathbb{Z}_+$, being the polynomials determined by (3.3).

Proof. With $f_1, f_2, f_3, f_4 \in \mathbb{C}[z]_q$, one can deduce from the results of section 7 that

$$(f_1(z) f_2(z)^*) * (f_3(z) f_4(z)^*) = f_1(z) B_{q,\alpha}(f_2(z)^* f_3(z)) f_4(z)^*. \tag{8.1}$$

An application of the results of section 5 to the bounded function $f_2(z)^* f_3(z)$ yields:

$$B_{q,\alpha}(f_2(z)^* f_3(z)) \underset{\alpha \to \infty}{\sim} (1 - q^{4\alpha}) \sum_{j \in \mathbb{Z}_+} q^{4\alpha \cdot j} p_j(\Box)(f_2(z)^* f_3(z)).$$

It remains to apply proposition 8.3 and the definition of $\widetilde{\Box}$. $\Box$

Remark 8.5. One can observe from proposition 2.5 that (8.1) is a q-analogue of relation (4.7) from [1].

Appendix. Overflowing vector systems

Unlike the main text where α was allowed to be an arbitrary positive number, let us assume now $\alpha \in \frac{1}{2}\mathbb{N}$.

Remind the notation $\widetilde{X}$ for the quantum principal homogeneous space, and $i : D(U)'_q \hookrightarrow D(\widetilde{X})_q$ for the canonical embedding of distribution spaces (see [10]).

Consider the embedding of vector spaces

$$i_\alpha : \mathrm{Pol}(\mathbb{C})_q \hookrightarrow D(\widetilde{X})'_q; \qquad i_\alpha : f \mapsto i(f) \cdot t_{12}^{-2\alpha-1}.$$

Equip $\mathrm{Pol}(\mathbb{C})_q$ with a new $U_q\mathfrak{sl}_2$-module structure given by $i_\alpha \xi f = \xi i_\alpha f$ for all $f \in \mathrm{Pol}(\mathbb{C})_q$, $\xi \in U_q\mathfrak{sl}_2$. Denote this $U_q\mathfrak{sl}_2$-module by $\mathrm{Pol}(\mathbb{C})_{q,\alpha}$. There exists an embedding $\mathbb{C}[z]_{q,\alpha} \hookrightarrow \mathrm{Pol}(\mathbb{C})_{q,\alpha}$.

The results of [10, section 6] imply

Proposition A.1. *The linear map $D(\widetilde{X})_q \to \mathbb{C}[z]_{q,\alpha}$ given by*

$$\psi \mapsto \int\limits_{\widetilde{X}_q} \tau_{12}^{*(-2\alpha-1)} \cdot (z\zeta^*; q^2)^{-1}_{2\alpha+1} \cdot \psi d\nu,$$

is a morphism of $U_q\mathfrak{sl}_2$-modules.

Proposition A.1 allows one to treat the function $\tau_{12}^{*(-2\alpha-1)} \cdot (z\zeta^*; q^2)^{-1}_{2\alpha+1}$ as a q-analogue of a coherent state in the sense of Perelomov [7].

Corollary A.2. *For all $\psi \in D(U)_q$*

$$P_{q,\alpha}\psi(z) = \int\limits_{U_q} (z\zeta^*; q^2)^{-1}_{2\alpha+1} \psi(\zeta) d\nu_\alpha(\zeta). \qquad (A.1)$$

Proof. Consider the integral operator

$$P : D(U)_q \to \mathbb{C}[z]_{q,\alpha}; \qquad P : \psi(z) \mapsto \int\limits_{U_q} (z\zeta^*; q^2)^{-1}_{2\alpha+1} \psi(\zeta) d\nu_\alpha.$$

It is a morphism of $U_q\mathfrak{sl}_2$-modules, as one can deduce from proposition A.1. The orthoprojection $P_{q,\alpha}$ is also a morphism of $U_q\mathfrak{sl}_2$-modules, due to the invariance of the scalar product in $H^2_{q,\alpha}$. It remains to use the relations $Pf_0 = 1 - q^{4\alpha}$, $P_{q,\alpha} f_0 = 1 - q^{4\alpha}$, together with the fact that f_0 generates the $U_q\mathfrak{sl}_2$-module $D(U)_q$ (see [9]). □

Remark A.3. (A.1) means that the distribution $(z\zeta^*; q^2)^{-1}_{2\alpha+1}$ is a reproducing kernel.

Let us find the kernel of the integral operator $P_{q,\alpha} \overset{\circ}{f} P_{q,\alpha}$. For $\overset{\circ}{f}, \psi \in D(U)_q$ one has by corollary A.2

$$P_{q,\alpha} \overset{\circ}{f} P_{q,\alpha} : \psi(z) \mapsto \int\limits_{U_q} K_q(\overset{\circ}{f}; z, z') \psi(z') d\nu_\alpha(z'),$$

with

$$K_q(\overset{\circ}{f}; z, z') = \int\limits_{U_q} (z\zeta^*; q^2)^{-1}_{2\alpha+1} \overset{\circ}{f}(\zeta) \cdot (\zeta z'^*; q^2)^{-1}_{2\alpha+1} d\nu_\alpha(\zeta).$$

Now an application of lemma 1.4 yields

$$K_q(\overset{\circ}{f}; z, z') = \frac{1-q^{4\alpha}}{1-q^2} \int\limits_{U_q} (z\zeta^*; q^2)^{-1}_{2\alpha+1} \overset{\circ}{f}(\zeta)(\zeta z'^*; q^2)^{-1}_{2\alpha+1}(1-\zeta\zeta^*)^{2\alpha+1} d\nu(\zeta) =$$

$$= \frac{1-q^{4\alpha}}{1-q^2} \int\limits_{U_q} (\zeta z'^*; q^2)^{-1}_{2\alpha+1}(1-\zeta\zeta^*)^{2\alpha}(z\zeta^*; q^2)^{-1}_{2\alpha+1} \cdot \overset{\circ}{f}(1-\zeta\zeta^*) d\nu(\zeta) =$$

$$= \frac{1-q^{4\alpha}}{1-q^2} \int\limits_{U_q} (1-\zeta\zeta^*)(\zeta z'^*; q^2)^{-1}_{2\alpha+1}(1-\zeta\zeta^*)^{2\alpha}(z\zeta^*; q^2)^{-1}_{2\alpha+1} \overset{\circ}{f} d\nu(\zeta).$$

Finally, use the relation $(1-\zeta\zeta^*)\zeta = q^2\zeta(1-\zeta\zeta^*)$ to obtain

Proposition A.4. $P_{q,\alpha} \overset{\circ}{f} P_{q,\alpha}$ *is an integral operator:*

$$P_{q,\alpha} \overset{\circ}{f} P_{q,\alpha}\psi(z) = \int\limits_{U_q} K_q(\overset{\circ}{f}; z, z')\psi(z') d\nu_\alpha(z'),$$

whose kernel is given by

$$K_q(\overset{\circ}{f}; z, z') = \frac{1-q^{4\alpha}}{1-q^2} \int\limits_{U_q} k_\zeta(q^2 z')^* \cdot k_\zeta(z) \overset{\circ}{f}(\zeta) d\nu(\zeta),$$

with $k_\zeta(z) = (1-\zeta\zeta^*)^{\alpha+1/2} \cdot (\zeta^* z; q^2)^{-1}_{2\alpha+1} \in D(U \times U)'_q$.

Proposition A.4 allows one to treat the distribution $k_\zeta(z)$ as a q-analogue of an overflowing vector system.

References

[1] F. Berezin, *General concept of quantization*, Commun. Math. Phys., **40** (1975), 153 – 174.

[2] V. Chari, A. Pressley. A Guide to Quantum Groups, Cambridge Univ. Press, 1995.

[3] N. Danford, J. T. Schwartz. Linear Operators, Part II, Interscience Publisher, N.-Y., London, 1963.

[4] G. Gasper, M. Rahman. Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.

[5] S. Klimec, A. Lesniewski, *A two-parameter quantum deformation of the unit disc*, J. Funct. Anal. **115**, (1993), 1 – 23.

[6] G. Nagy, A. Nica. *On the "quantum disc" and a "non-commutative circle"*, in: Algebraic Methods on Operator Theory, R. E. Curto, P. E. T. Jorgensen (eds.), Birkhauser, Boston, 1994, p. 276 – 290.

[7] A. Perelomov. Generalized Coherent States and their Applications. Berlin, Heidelberg, New York: Springer, 1986.

[8] D. Shklyarov, S. Sinel'shchikov, L. Vaksman. On function theory in quantum disc: integral representations, E-print: math.QA/9808015

[9] D. Shklyarov, S. Sinel'shchikov, L. Vaksman. On function theory in quantum disc: covariance, E-print: math.QA/9808037

[10] D. Shklyarov, S. Sinel'shchikov, L. Vaksman. On function theory in quantum disc: invariant kernels, E-print: math.QA/9808047

[11] D. Shklyarov, S. Sinel'shchikov, L. Vaksman. On function theory in quantum disc: q-differential equations and Fourier transform, E-print: math.QA/9809002

[12] S. Sinel'shchikov, L. Vaksman. *On q-analogues of Bounded Symmetric Domains and Dolbeault Complexes*, Mathematical Physics, Analysis and Geometry; Kluwer Academic Publishers, V.1, No.1, 1998, 75–100, E-print: q-alg/9703005.

[13] A. Unterberger, H. Upmeier, *The Berezin transform and invarian differential operators*, Comm. Math. Phys., **164** (1994), 563 – 598.